Quantum Damped Harmonic Oscillator (Joseph G. Depp, May 1999 updated 2023)

The quantum damped harmonic oscillator (QDHO) combines concepts from the classical damped harmonic oscillator and the quantum harmonic oscillator. Several approaches to the QDHO have been suggested including non-linear approaches and time-dependent Hamiltonians. In the 1940's a time-dependent Hamiltonian was published by Caldirola [1] and Kanai [2]. The Hamiltonian has become known as the Caldirola-Kanai (CK) Hamiltonian. Renewed interest in the CK Hamiltonian was expressed by Papadopoulos [3] in 1973. A full solution to the CK Hamiltonian was published by Yeon [4] in 1987 (See the footnote at the end of this section for more on the solution by Yeon.). The QDHO was revisited by several authors in the late 1990s including Huang [5] and Papadopoulos [6]. However, to date it has not been recognized that the general partial wave solution can be brought into a much simpler form that can be used for a variety of applications. It is the purpose of this paper to present the simpler solution and apply it to the transition from quantum to classical damped harmonic oscillator.

We begin with the CK Hamiltonian:

$$H = -exp(-2\gamma t)\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{m\omega_0^2}{2}exp(2\gamma t)x^2 \tag{1}$$

In this presentation we have chosen to follow the notation and development of Papadopoulos [3]. We can simplify the development a bit by setting

$$x = \left(\frac{\hbar}{m\omega_0}\right)^{1/2}\xi \tag{2}$$

The CK Hamiltonian becomes

$$H = \frac{\hbar\omega_0}{2}\left[-exp(-2\gamma t)\frac{\partial^2}{\partial\xi^2} + exp(2\gamma t)\xi^2\right] \tag{3}$$

The propagator for this Hamiltonian is

$$K(\xi, t|\xi', 0) = \left[\frac{\omega\, exp(\gamma t)}{2\pi i\omega_0\, sin(\omega t)}\right]^{1/2} exp[iS(\xi, t|\xi', 0)] \tag{4}$$

where

$$S(\xi, t|\xi', 0) = \frac{\gamma}{2\omega_0}[\xi'^2 - exp(2\gamma t)\xi^2] + \frac{\omega}{2\omega_0\, sin(\omega t)}\{cos(\omega t)[\xi'^2 + exp(2\gamma t)\xi^2] - 2\xi\xi'\, exp(\gamma t)\} \tag{5}$$

and

$$\omega^2 = \omega_0^2 - \gamma^2 \tag{6}$$

Note that omega can be real, zero, or imaginary depending on the values of omega zero and gamma.

The partial waves, $\Psi_L(\xi,t)$, are obtained by propagating the partial waves, $\Phi_L(\xi)$, of the quantum harmonic oscillator using the propagator as follows:

$$\Psi_L(\xi,t) = \int_{-\infty}^{+\infty} K(\xi,t|\xi',0)\, \Phi_L(\xi')d\xi' \tag{7}$$

The partial waves, $\Phi_L(\xi)$, can be found in most textbooks on quantum mechanics.

$$\Phi_L(\xi) = \frac{1}{\pi^{1/4}}(2^L L!)^{-1/2}\, exp(-\xi^2/2)\, H_L(\xi) \tag{8}$$

$H_L(\xi)$ is the Hermite polynomial of order L. By using the generating function for the Hermite polynomials the integral can be performed for all partial waves at the same time. The resulting partial waves are given by:

$$\Psi_L(\xi,t) = G^{1/2}\Phi_L(G\xi)\, exp[i(L+1/2)(\varepsilon-\pi/2)]\, exp[-i(b+aG^2)\xi^2] \tag{9}$$

G, a, b, and ε are functions of time only.

$$a(t) = \left\{\frac{\gamma}{2\omega_0} + \frac{\omega\, cos(\omega t)}{2\omega_0\, sin(\omega t)}\right\} \tag{10}$$

$$b(t) = \left\{\frac{\gamma}{2\omega_0} - \frac{\omega\, cos(\omega t)}{2\omega_0\, sin(\omega t)}\right\} exp(\,2\gamma t) \tag{11}$$

$$G(t) = \frac{\omega\, exp(\gamma t)}{2\omega_0\, sin(\omega t)[1/4+a^2]^{1/2}} \tag{12}$$

$$\varepsilon(t) = tan^{-1}(2a) \tag{13}$$

The equations above are valid for all ω, also a, b and G are real for all ω.
Moreover, G, ε, and $(b + aG^2)$ are finite for all times including t=0.

First, we note that, at t=0, G=1, $\varepsilon = \pi/2$, and $(b + aG^2) = 0$ so that

$$\Psi_L(\xi,0) = \Phi_L(\xi) \tag{14}$$

as it should.

Next, we note that the phase angle, $(b + aG^2)\xi^2$ can be greatly simplified after significant algebraic manipulation:

$$(b + aG^2)\xi^2 = \frac{\gamma\omega_0}{\omega^2}(\sin \omega t)^2(G^2\xi^2) \tag{15}$$

allowing us to make a change of variable, $\bar{\xi} = G\xi$, and write (9) as:

$$\Psi_L\left(\bar{\xi},t\right) = G^{1/2}\Phi_L\left(\bar{\xi}\right) exp[i(L+1/2)(\varepsilon-\pi/2)]\, exp\left[-i\frac{\gamma\omega_0}{\omega^2}(\sin \omega t)^2\bar{\xi}^2\right] \tag{16}$$

Equation (16) shows that the QDHO can be represented as standard quantum harmonic oscillator with the displacement replaced by a time dependent scaling factor and a time dependent phase factor.

It is apparent that the partial waves are orthonormal for all times, t.

$$\int_{-\infty}^{+\infty} \Psi_M^*(\xi,t)\Psi_L(\xi,t)d\xi = \exp\{i(L-M)[\varepsilon(t)-\pi/2]\}\int_{-\infty}^{+\infty} \Phi_M(\bar{\xi})\Phi_L(\bar{\xi})\,d\bar{\xi} \tag{17}$$

$$= \delta_{ML}$$

The full wavefunction $\Psi(\xi,t)$ is given by a linear combination of partial waves:

$$\Psi(\xi,t) = \sum_{L=0}^{\infty} A_L\, exp(\, i\phi_L)\,\Psi_L(\xi,t) \tag{18}$$

$A_L$and $\phi_L$are arbitrary real constants subject to the constraints that

$$A_L \geq 0 \tag{19}$$

and

$$\sum_{L=0}^{\infty} A_L^2 = 1 \tag{20}$$

We would like to compute the expectation value of ξ defined as

$$\langle\xi\rangle = \int_{-\infty}^{+\infty} \Psi^*(\xi,t)\xi\,\Psi(\xi,t)d\xi \tag{21}$$

The integrals can be performed with the substitution from (16) and by using the recursion relations for the Hermite polynomials. After a bit of manipulation, we obtain:

$$\langle\xi\rangle = \left\{C\left[\frac{\omega_0}{\omega}\sin(\omega t)\right] + D\left[\frac{\gamma}{\omega}\sin(\omega t) + \cos(\omega t)\right]\right\}\exp(-\gamma t) \tag{22}$$

We see that the expectation value of ξ is just the equation that we would have gotten for a classical damped harmonic oscillator. The constant D is the position at time t=0 and the constant C is related to the initial velocity. It is $\frac{1}{\omega_0}\frac{d\langle\xi\rangle}{dt}$ at t=0.

The constants C and D can be expressed in terms of the coefficients of the full wavefunction by:

$$C = -\sqrt{2}\sum_{L=0}^{\infty} A_{L+1}\,A_L\sqrt{L+1}\,sin(\,\phi_L - \phi_{L+1}) \tag{23}$$

$$D = +\sqrt{2}\sum_{L=0}^{\infty} A_{L+1}\,A_L\sqrt{L+1}\,cos(\,\phi_L - \phi_{L+1}) \tag{24}$$

Equations (23) and (24) provide the link that we seek between the QDHO and the classical damped harmonic oscillator.

The initial quantum energy is just the energy of a quantum harmonic oscillator:

$$E_q = \hbar\omega_0(1/2 + \sum_{L=0}^{\infty} A_L^2 L) \tag{25}$$

The initial value of the classical energy is the mechanical energy:

$$E_{cl} = \frac{\hbar\omega_0}{2}(C^2 + D^2) \tag{26}$$

Note that for C and D to be non-zero there must be at least one pair of adjacent partial waves to allow the emission of one quantum of energy. It is possible to envision a QDHO wavefunction that has no classical analog. Such a wavefunction is either symmetric or anti-symmetric. It has no initial displacement and no initial velocity (and no mechanical energy) nevertheless it does carry quantum energy that cannot be dissipated.

Footnote on Yeon solution: Unfortunately, there is an error in one of the equations in Yeon that has caused some confusion. In Yeon equation (6) the normalization function should be:

$$N = \left[\frac{m\omega}{\pi\hbar}\right]^{1/4} \frac{\exp(-\frac{\gamma t}{4})}{\left[\varsigma(t)\sin(\omega t)\right]^{1/2}}$$

With this expression it can be seen that:

$$N^2 = \frac{D}{\sqrt{\pi}}$$

where D is also defined in Yeon equation (6). This relationship is consistent with the results above. (Note that there is a difference in definitions of gamma between Yeon and Papadopoulos such that

$$\gamma_{Yeon} = 2\gamma_{Papadopoubs}$$

References

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